ASYMPTOTIC PATTERN OF FLOW PAST BODIES OF REVOLUTION IN A SONIC STREAM OF VISCOUS AND HEAT-CONDUCTING GAS

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The first attempt at simplification of the Navier-Stokes equations for describing two-dimensional steady transonic flows of a perfect gas (*) is evidently attributable to Liepmann, Ashkenas and Cole (see, for example,[1]). The asymptotic equations obtained by them were derived by other methods in papers by Sternberg [2], Sichel [3 and 4] and Szaniawski [5 and 6]. Moreover, these papers mentioned several examples of flows with velocities close to the speed of sound, on the structure of which viscosity and heat conduction can exert a substatial influence. Relying on the Liepmann-Ashkenas-Cole equation, the author together with Shefter showed [7] that the asymptotic pattern of flow past bodies of revolution of a stream of viscous, heat-conducting gas which is sonic at infinity, must differ qualitatively from that which is given by the solution of the equations of an ideal gas [8 to 10]. This conclusion is obtained unexpectedly, since for an incompressible fluid the behavior of the solutions of the equations of Euler and Navier-Stokes are in the first approximation identical, if they are considered at points located outside the vortex trail and sufficiently far removed from the immersed body.

In the present paper is derived a detailed analysis of the Navier-Stokes equations under the assumption that the particle velocity of the gas is close to the speed of sound in the entire region of flow. Besides the nonlinear equation of Liepmann-Ashkenas-Cole, we derive the much simpler linear equation which is valid if the field of flow is determined on the basis of viscosity and heat conduction. This equation is then applied to the study of the asymptotic laws of decay of disturbances with distance from the body of revolution immersed in a stream which is sonic at infinity. It is established that the width l of the zone in which the values of the gas parameters differ appreciably from the corresponding values in the free stream is proportional to $r^{2/3}$, where r is the distance from the axis of symmetry. The difference between the local value of the Mach number M and unity is universely proportional to $r^{4/3}$, and the angle ϑ between the velocity vector and the direction of motion of the undisturbed stream is inversely proportional to $r^{5/3}$. As was shown by Guderley, Yoshihara and Barish [8 and 9],

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*) The term "perfect" denotes a gas governed by Clapeyrons's equation of state: the name "ideal" will relate to a gas devoid of viscosity and heat conduction.

Fal'kovich and Chernov [10], in the analogous problem with vanishing coefficients of viscosity and heat conduction, $l \sim r^{4/7}$, $M - 1 \sim r^{-4/7}$ and $\vartheta \sim r^{-6/7}$. The presence of the dissipative factors leads to the disappearance of the shock wave and to a continuous character of the flow.

1. Derivation of the approximate equations. Let x and r denote cylindrical space coordinates, v_x and v_r the components of the velocity vector along the x- and r-axes, ρ the density, p the pressure, s the specific entropy, T the temperature, λ_1 the coefficient of viscosity, λ_2 the second coefficient of viscosity, k the coefficient of heat conductivity. Assuming the field of flow is symmetric with respect to the x-axis, we take the Navier-Stokes equations of continuity and heat transfer in the form [11]

$$\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_r}{\partial r} + \frac{\rho v_r}{r} = 0 \tag{1.1}$$

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_r \frac{\partial v_x}{\partial r}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\lambda_1 \frac{\partial v_x}{\partial x} + \left(\lambda_2 - \frac{2}{3}\lambda_1\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)\right] + \frac{\partial}{\partial r}\left[\lambda_1\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)\right] + \frac{\lambda_1}{r}\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right) \tag{1.2}$$

$$\rho\left(v_x \frac{\partial v_r}{\partial x} + v_r \frac{\partial v_r}{\partial r}\right) = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial x}\left[\lambda_1\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)\right] + \frac{\partial}{\partial r}\left[2\lambda_1 \frac{\partial v_r}{\partial r} + \left(\lambda_2 - \frac{2}{3}\lambda_1\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)\right] + \frac{2\lambda_1}{r}\left(\frac{\partial v_r}{\partial r} - \frac{v_r}{r}\right) \tag{1.3}$$

$$\rho T\left(v_x \frac{\partial s}{\partial x} + v_r \frac{\partial s}{\partial r}\right) = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial r}\left(k \frac{\partial T}{\partial r}\right) + \frac{k}{r}\frac{\partial T}{\partial r} + 2\lambda_1\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)^2 + \left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{v_r}{r}\right)^2\right] + \left(\lambda_2 - \frac{2}{3}\lambda_1\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)^2 \tag{1.4}$$

In order to complete the system, we need to add two further equations which relate the thermodynamic quantities ρ, p, s and T. In what follows it is convenient to take for the independent parameters the density and pressure, whilst the specific entropy and temperature are expressed in terms of them. The increment of entropy

$$ds = \left(\frac{\partial s}{\partial p}\right)_\rho dp + \left(\frac{\partial s}{\partial \rho}\right)_p d\rho = \left(\frac{\partial s}{\partial p}\right)_\rho \left(dp - a^2\, d\rho\right) \qquad \left(a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s\right)$$

Here a is the adiabatic velocity of sound. We introduce also the specific heats at constant pressure c_p and constant volume c_v and the coefficient of thermal expansion α. Using the relations of reciprocity, we shall have [12]

$$\left(\frac{\partial s}{\partial p}\right)_\rho = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_s = \frac{c_v}{T}\left(\frac{\partial T}{\partial p}\right)_\rho \tag{1.5}$$

The temperature differential is

$$dT = \left(\frac{\partial T}{\partial p}\right)_\rho dp + \left(\frac{\partial T}{\partial \rho}\right)_p d\rho = \left(\frac{\partial T}{\partial p}\right)_\rho \left[dp - \left(\frac{\partial p}{\partial \rho}\right)_T d\rho\right]$$

Between the isothermal and adiabatic velocities of sound there exists the simple relation [12]

$$\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{1}{\varkappa}\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{a^2}{\varkappa} \qquad \left(\varkappa = \frac{c_p}{c_v}\right) \tag{1.6}$$

It remains now to calculate the partial derivative $(\partial T / \partial p)_\rho$. For this purpose we use the well-known formula from analysis

$$\left(\frac{\partial T}{\partial p}\right)_\rho \left(\frac{\partial p}{\partial \rho}\right)_T \left(\frac{\partial \rho}{\partial T}\right)_p = -1$$

Hence we find without difficulty that

$$\left(\frac{\partial T}{\partial p}\right)_\rho = \frac{\varkappa}{\alpha \rho a^2} \qquad \left(\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p\right) \tag{1.7}$$

Using Equations (1.5) to (1.7) for the increments of specific entropy and temperature, we derive the relations

$$ds = \frac{c_p}{\alpha T \rho a^2}(dp - a^2 d\rho), \qquad dT = \frac{1}{\alpha \rho a^2}(\varkappa dp - a^2 d\rho) \tag{1.8}$$

Equations (1.1) to (1.4), together with the relations (1.8) form a complete system. In the analysis of it we make the assumption that the values of all the gas parameters in the region of space under consideration differ only slightly from the corresponding values in the free stream. We shall assume that the undisturbed flow is uniform and steady, with the particle velocity equal in magnitude to the sound velocity and directed along the x-axis. The values of the gas parameters in the undisturbed state will be distinguished by an asterisk, and the characteristic length in the direction of the x-axis will be denoted by L. With regard to the perturbations of density, pressure, temperature and sound velocity, we shall assume that they have the same order of smallness as the longitudinal component of the velocity vector. Passing to dimensionless variables, we write

$$\begin{gathered} x = Lx', \quad r = \frac{L}{\Delta} r', \quad v_x = a_* (1 + \varepsilon v_x'), \quad v_r = \varepsilon \Delta a_* v_r' \\ \rho = \rho_* (1 + \varepsilon \rho'), \quad p = p_* (1 + \varepsilon p'), \quad a = a_* (1 + \varepsilon a') \end{gathered} \tag{1.9}$$

Here ϵ and Δ are numerical parameters which are small in magnitude compared with unity. As a result of substituting the relations (1.9) in the system of equations (1.1) to (1.4) and (1.8) we obtain three dimensionless coefficients

$$N_{\text{Re}1} = \frac{\rho_* a_* L}{\lambda_1}, \qquad N_{\text{Re}2} = \frac{\rho_* a_* L}{\lambda_2}, \qquad N_{\text{Pe}} = \frac{\rho_* a_* L c_p}{k}$$

On the same principle, we shall assume that reciprocal values of these numbers have the same order and are small compared with unity. In the derivation of the approximate equations in all relations we shall retain only the main terms, neglecting terms having a higher order of smallness. Hence in Equations (1.2) to (1.4) the coefficients of viscosity λ_1, λ_2 and of heat conductivity k can be set constant and equal to their values in the free stream. Concerning the choice of order of the transverse component of the velocity vector in Formulas (1.9), this is justified as a result of the subsequent analysis.

After linearization of the equation of continuity we obtain (*)

$$\partial v_x / \partial x + \partial \rho / \partial x = 0$$

From the projection of the Navier-Stokes equation on the x-axis it follows that

$$\partial v_x / \partial x + p_* \rho_*^{-1} a_*^{-2} \partial p / \partial x = 0$$

Integration of the two last equations gives Formulas

$$p_* \rho_*^{-1} a_*^{-2} p = \rho = - v_x \tag{1.10}$$

which express the fact that in the approximation under consideration the compression of the gas is accomplished adiabatically and the integral of Bernoulli is valid over the whole flow. This deduction is the direct result of the assumption concerning the smallness of the derivation of the parameters of the medium in the field of perturbations from the corresponding quantities in the equilibrium state, and also the assumption concerning the large values of the Reynolds number, compared with unity.

The postulated assumptions lead further to Equation

$$\frac{\partial v_x}{\partial r} = \frac{\partial v_r}{\partial x} \tag{1.11}$$

following from the projection of the Navier-Stokes equation on the r-axis and Formulas (1.10) and the postulated absence of vortices from the flow. Accordingly, by simplification of the three first equations of the system (1.1) to (1.4) we obtain expressions characterizing the motion of ideal media. The influence of dissipative factors must be taken into account in considering the equations of heat transfer. It is necessary in the preliminary transformation to exclude quantities of the first order of smallness, related to the transfer of mass and momentum of the material. Passing in Equation (1.4) from entropy and temperature to pressure and density by means of Formulas (1.8) and combining the expression so obtained with Equation (1.1) and (1.2), we arrive at the required relation

$$v_x \left[(v_x^2 - a^2) \frac{\partial \rho}{\partial x} - \rho v_r \frac{\partial v_x}{\partial r} \right] + v_r \left(\frac{\partial p}{\partial r} - a^2 \frac{\partial \rho}{\partial r} \right) + v_x^2 \left(v_r \frac{\partial \rho}{\partial r} + \rho \frac{\partial v_r}{\partial r} + \frac{\rho v_r}{r} \right) = \\ = \frac{\alpha a^2}{c_p} L_1(k, \lambda_1, \lambda_2) - v_x L_2(\lambda_1, \lambda_2) \tag{1.12}$$

Here we denote by $L_1(k, \lambda_1, \lambda_2)$ the right-hand side of Equation (1.4), and by $L_2(\lambda_1, \lambda_2)$ the right-hand side of Equation (1.2) without the first term. As is shown by Formulas (1.9), the difference $v_x^2 - a^2$ is proportional to the parameter ϵ. In the approximation under consideration

$$da = \left(\frac{\partial a}{\partial \rho_*} \right)_s d\rho = \frac{(m_* - 1) a_*}{\rho_*} d\rho \qquad \left(m_* = \frac{1}{2\rho_*^3 a_*^2} \left(\frac{\partial^2 p}{\partial V_*^2} \right)_s, \ V = \frac{1}{\rho} \right)$$

Using the latter relations and substituting Formulas (1.9) into Equation (1.12), we obtain

$$2 m_* \epsilon v_x \frac{\partial v_x}{\partial x} - \Delta^2 \left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} \right) = \frac{1}{N_{\mathrm{Re}}} \left(1 + \frac{\varkappa - 1}{N_{\mathrm{Pr}}} \right) \frac{\partial^2 v_x}{\partial x^2} \tag{1.13}$$

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*) Primes on dimensionless variables will now and henceforth be omitted.

Equations (1.11) and (1.13) form a complete system, and moreover the analysis of the latter enables us to distinguish the different cases which can occur in the study of transonic flows of viscous heat conducting gas. The overall Reynolds' number N_{Re} appearing in Equation (1.13) is related to the so-called "longitudinal viscosity"

$$\frac{1}{N_{\mathrm{Re}}} = \frac{4}{3}\frac{1}{N_{\mathrm{Re1}}} + \frac{1}{N_{\mathrm{Re2}}}$$

and the Prandtl number N_{Pr} is simply the ratio of the Péclet number N_{Pe} to the Reynolds' number N_{Re}. The orders of the Péclet and Reynolds' numbers are by assumption the same, so that the Prandtl number is of order unity. We notice that the terms in Equation (1.4), related to the dissipation of energy on account of viscous forces, do not influence the expression on the right-hand side of Equation (1.13). Let us proceed to the complete classification of the flows.

1) Suppose that all the terms in Equation (1.13) have the same order of magnitude. This case was apparently first considered by Liepmann, Ashkenas and Cole [1], who in the simplification of the initial Navier-Stokes equations tended to retain not only the basic terms related to the presence of the dissipative factors, but also the main nonlinear term which is obtained in the theory of flow of an ideal gas.

2) If $\Delta^2 \ll \varepsilon \sim N_{\mathrm{Re}}^{-1}$, then we have

$$2m_* \varepsilon v_x \frac{\partial v_x}{\partial x} = \frac{1}{N_{\mathrm{Re}}}\left(1 + \frac{\varkappa - 1}{N_{\mathrm{Pr}}}\right)\frac{\partial^2 v_x}{\partial x^2}$$

This equation describes, in particular, the structure of shock waves.

3) If $\Delta^2 \sim \varepsilon$ whilst $N_{\mathrm{Re}}^{-1} \ll \varepsilon$, then the effect of viscosity and heat conduction can be neglected. Setting for simplicity $2m_*\varepsilon = \Delta^2$, we arrive at Kármán's equation [13]

$$-v_x \frac{\partial v_x}{\partial x} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0 \qquad (1.14)$$

4) Finally, let $\Delta^2 \sim N_{\mathrm{Re}}^{-1}$, whilst $\varepsilon \ll N_{\mathrm{Re}}^{-1}$. In this case the nonlinear term in Equation (1.13) can be neglected in comparison with the rest. Assuming that

$$\Delta^2 = \frac{1}{N_{\mathrm{Re}}}\left(1 + \frac{\varkappa - 1}{N_{\mathrm{Pr}}}\right)$$

we have

$$\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0 \qquad (1.15)$$

In so far as the relation (1.11) is the condition of existence of a velocity potential, i.e. a function $\varphi(x, r)$ whose differential $d\varphi = v_x dx + v_r dr$, then Equation (1.15) can be put in the form

$$\frac{\partial^3 \varphi}{\partial x^3} + \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} = 0 \qquad (1.16)$$

Equation (1.16) is of parabolic type. Up to now it has not been encountered either in physical or in technical problems. Certain properties of the plane analogy of this equation, in which the term $r^{-1}\partial\varphi/\partial r$ does not appear, were studied recently by Dezin [14 and 15]. The fundamental advantage of Equation (1.16) in comparison with the equation of Liepmann-Ashkenas-Cole consists in its linearity. It can serve as a mathematical model for the study of the motion of viscous, heat conducting gas with transonic velocities, if the structure of the flow is governed fundamentally by the influence of the dissipative factors.

2. Streamline flow past a finite body. Let us consider the velocity field around a body with a circular cross section, immersed in a stream which is sonic at infinity.

The asymptotic laws of decay of disturbances at great distances from the body were studied by Guderley, Yoshihara and Barish [8 and 9], who in the solution of the boundary value problem made use of numerical methods of integration of ordinary differential equations. The exact solution of this problem was given subsequently by Fal'kovich and Chernov [10]. The fundamental assumption, on which were based the results of papers [8 to 10], consisted in that the gas was devoid of viscosity and heat conductivity. However, all media existing in reality can conduct heat, and viscous forces occur in them. As a rule, the influence of dissipative factors arises significantly only in the boundary layer and in the vortex trail behind the body, and it determines the structure of the shock waves. In calculation of the field of flow far from the body it is usually assumed that the influence of viscosity and heat conduction can be neglected. If the fluid is incompressible, then Finn proved rigorously that at those points located outside the vortex trail and sufficiently far from the body the asymptotic solutions of Euler's equations serve simultaneously as asymptotic solutions of the Navier-Stokes equations. The structure of the wake in the first approximation is correctly determined by the linearized equations of Prandtl for the boundary layer (*).

In a joint paper the author and Shefter showed that in the study of the asymptotic laws of decay of disturbances caused by a finite body of revolution in a stream which is sonic at infinity, the influence of viscosity and heat conductivity cannot be neglected [7]. On substituting the solution found in the papers [8 to 10] into Equation (1.13), the term standing in its right-hand side turns out to be greater in magnitude than the terms on the left-hand side, if the values of the coordinate r tend to infinity. Accordingly, for correct determination of the laws of decay of disturbances in a real gas it is necessary to make use, not of Kármán system of equations (1.11),(1.14), but of the system of equations (1.11),(1.15) derived above, or of the third order equation (1.16).

In the following investigation the system of equations for the flow velocity components with respect to the coordinates turns out to be more convenient than the potential equation.

It is not difficult to verify that the system of equations (1.11), (1.15) is invariant with respect to the continuous two-parameter group of similarity transformations

$$x \to \alpha x \qquad r \to \alpha^{3/2} r, \qquad v_x \to \alpha^{-3n/2} v_x, \qquad v_r \to \alpha^{-(3n+1)/2} v_r$$

Hence follows the existence of self-similar solutions of it, having the form

$$v_x = r^{-n} f(\xi), \qquad v_r = r^{-(3n+1)/3} g(\xi), \qquad \xi = x r^{-2/3} \tag{2.1}$$

Making use of the solution (2.1) we find how the components of the velocity vector of the gas particles decrease with distance. The problem reduces, evidently, to finding the value of the parameter n for which the function $v_r(x,r)$ vanishes along the whole r-axis with the exception of the origin of coordinates $x = r = 0$. At infinitely distant points of the stream both

*) Finn's results are available in his review paper [16].

components of the velocity vector must vanish, which leads to the requirement that $n > 0$. The results of [7] show that actually the inequality $n > {}^2/_3$ should be satisfied.

The substitution of Formulas (2.1) into the initial system (1.11), (1.15) gives

$$\frac{2}{3}\xi\frac{df}{d\xi}+\frac{dg}{d\xi}=-nf, \qquad \frac{d^2f}{d\xi^2}=\frac{2}{3}\xi\frac{dg}{d\xi}+\left(n-\frac{2}{3}\right)g \tag{2.2}$$

Eliminating from Equations (2.2) the function $g(\xi)$, we obtain for $f(\xi)$ an ordinary differential equation of the third order

$$\frac{d^3f}{d\xi^3}+\frac{4}{9}\xi^2\frac{d^2f}{d\xi^2}+\frac{4}{3}\left(n+\frac{1}{3}\right)\xi\frac{df}{d\xi}+n^2f=0 \tag{2.3}$$

after the solution of which the quantity $g(\xi)$ is found by means of Equation

$$g=\frac{3}{3n-2}\left(\frac{d^2f}{d\xi^2}+\frac{4}{9}\xi^2\frac{df}{d\xi}+\frac{2}{3}n\xi f\right) \tag{2.4}$$

When $r \to 0$, the value of the self-similar coordinate ξ in absolute magnitude increases without limit. Let us write down the asymptotic expansions for three linearly independent solutions of Equation (2.3) when $|\xi|\to\infty$.

The first of them is

$$f=a_1|\xi|^{-3n/2}[1+{}^{27}/_{32}n(n+{}^2/_3)(n+{}^4/_3)\xi^{-3}+\ldots] \tag{2.5}$$

The second of the required solutions we shall take in the form

$$f=a_2|\xi|^{-3n/2}\ln|\xi|+\ldots \tag{2.6}$$

whilst the third linearly independent solution of Equation (2.3) we shall write as

$$f=a_3|\xi|^{3(n-1)}\exp(-{}^4/_{27}\xi^3)+\ldots \tag{2.7}$$

Using Equations (2.1) and (2.4) it is easy to see that the asymptotic expansions corresponding to Formula (2.5) for the functions $v_x(x,r)$ and $v_r(x,r)$, as $r \to 0$, start with the terms

$$v_x=a_1|x|^{-3n/2}+\ldots, \qquad v_r=-{}^9/_8\,n(n+{}^2/_3)a_1|x|^{-(3n+4)/2}r+\ldots$$

The longitudinal component of the perturbation velocity of the stream, calculated according to the solution (2.6), tends to infinity like $\ln r$, whilst the transverse component tends like $1/r$. As to the solution (2.7), it gives expressions for both components of the gas particle velocity which contain the common factor $\exp(-4x^3/27r^2)$. Therefore, the use of solution (2.7) with negative values of x turns out to be impossible, whilst on the other hand, with positive values of x the corresponding disturbances decay very quickly close to the axis of symmetry.

The results obtained make it possible to formulate the problem on the eigenvalues for the ordinary linear differential equation (2.3): it is required to find the value of the parameter n for which the integral of this equation is defined by the expansion (2.5) when $\xi \to -\infty$, and satisfies the condition

$$\frac{3n-2}{3}|\xi|^{(3n-2)/2}g(\xi)\to 0 \tag{2.8}$$

if $\xi \to +\infty$. The condition (2.8) is equivalent to the requirement that the constant a_2 in expansion (2.6) should vanish.

The first eigenvalue n is equal to $4/3$. Simple verification serves to confirm this assertion. In this case the second of the initial equations (2.2) integrates to give

$$df \,/\, d\xi = {}^2/_3 \xi g + b$$

The constant b must be set equal to zero in order to satisfy the condition (2.8). For the function $f(\xi)$ we obtain

$$\xi \frac{d^2 f}{d\xi^2} + \left(\frac{4}{9}\xi^3 - 1\right)\frac{df}{d\xi} + \frac{8}{9}\xi^2 f = 0 \qquad (2.9)$$

From this differential equation by differentiation we obtain (2.3). The asymptotic expansion for the first linearly independent solution of the stipulated equation when $|\xi| \to \infty$ is given by Formula (2.5), and for the second by Formula (2.7). Since the logarithmic terms in the asymptotic expansions of both solutions are absent, the equation $n = 4/3$ actually makes it possible to obtain the solution of the boundary value problem formulated above.

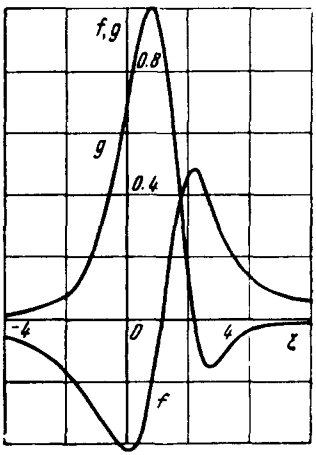

Fig. 1

In order to find the exact expression for the first eigenfunction we effect the substitution of the independent variable $\eta = -{}^4/_{27}\xi^3$ in Equation (2.9), as a result we have

$$\eta \frac{d^2 f}{d\eta^2} + \left(\frac{1}{3} - \eta\right)\frac{df}{d\eta} - \frac{2}{3} f = 0 \qquad (2.10)$$

The equation obtained represents in canonical form the so-called confluent hypergeometric equation [17]. Using the standard notation for the confluent hypergeometric functions, we obtain for the general solution

$$f = c_1 \Phi ({}^2/_3, {}^1/_3; \eta) + c_2 \eta^{2/3} \Phi ({}^4/_3, {}^5/_3; \eta) \qquad (2.11)$$

It remains to find the relation between the constants c_1 and c_2. For this purpose we use the asymptotic representation of the hypergeometric functions $\eta \to +\infty$. We have [17]

$$f = \eta^{1/3} e^{\eta} G\left(-\frac{1}{3}, \frac{1}{3}; \eta\right)\left[\frac{\Gamma(1/3)}{\Gamma(2/3)} c_1 + \frac{\Gamma(5/3)}{\Gamma(4/3)} c_2\right] + \ldots$$

where Γ denotes Euler's gamma function, whilst $G(-{}^1/_3, {}^1/_3; \eta)$ is a series in inverse powers of η, and $G(-{}^1/_3, {}^1/_3; \eta) \to 1$ as $\eta \to +\infty$. In order to obtain the solution tending to zero at infinity it is necessary to set

$$\frac{c_2}{c_1} = -\frac{\Gamma(1/3)\,\Gamma(4/3)}{\Gamma(2/3)\,\Gamma(5/3)} = -\frac{\Gamma^2(1/3)}{2\Gamma^2(2/3)}$$

Now Formula (2.11) is transformed into the form

$$f = c_1\left[\Phi\left(\frac{2}{3}, \frac{1}{3}; \eta\right) - \frac{\Gamma^2(1/3)}{2\Gamma^2(2/3)} \eta^{2/3} \Phi\left(\frac{4}{3}, \frac{5}{3}; \eta\right)\right] \qquad (2.12)$$

The linear combination of hypergeometric functions included in square brackets is proportional to the so-called Ψ-function [17], hence when $\eta \to +\infty$ and $\xi \to -\infty$ we deduce that

$$f = 3\,\frac{\Gamma(^2/_3)}{\Gamma(^1/_3)}\,c_1\eta^{-^2/_3} + \ldots = \frac{27\Gamma(^2/_3)}{2\sqrt[3]{2}\,\Gamma(^1/_3)}\,c_1\xi^{-2} + \ldots \tag{2.13}$$

Making use of the asymptotic expansion of the hypergeometric functions for large negative values of the argument [17], we find for the solution (2.12) when $\eta \to -\infty$ and $\xi \to +\infty$

$$f = -\frac{2}{3}\,\frac{\Gamma(^1/_3)}{\Gamma(^2/_3)}\,c_1\eta^{-^2/_3} + \ldots = -\frac{3\Gamma(^1/_3)}{\sqrt[3]{2}\,\Gamma(^2/_3)}\,c_1\xi^{-2} \tag{2.14}$$

A graph of the function $f(\xi)$, describing the variation of the gas particle velocity along a streamline, is shown in Fig.1. In the calculations we used $c_1 = -2\sqrt[3]{2}\Gamma(^1/_3)[27\Gamma(^2/_3)]^{-1}$. To the approximation under consideration $v_x \sim M-1$, where M denotes the local Mach number, hence $f(\xi) \sim r^{^4/_9}(M-1)$. Let us now introduce the angle ϑ between the velocity vector and the x-axis. The function $g(\xi) \sim r^{^5/_9}\vartheta$, and its graph is also shown in Fig.1. In accordance with Formulas (2.13) and (2.14), the function $f(\xi)$ passes once through zero and has different signs when $\xi \to -\infty$ and $\xi \to +\infty$. As to the function $g(\xi)$, it also has different signs for large negative and positive values of the argument, but whilst its maximum value is equal to 1.01, its minimum value is equal to -0.16. The exact expression for $g(\xi)$ can be found with the help of the rule for differentiation of confluent hypergeometric functions [17

$$g = 2\sqrt[3]{2}\,c_1\left[\eta^{^1/_3}\Phi\left(\frac{5}{3},\frac{4}{3};\eta\right) - \frac{\Gamma^2(^1/_3)}{6\Gamma^2(^2/_3)}\,\Phi\left(\frac{4}{3},\frac{2}{3};\eta\right)\right] \tag{2.15}$$

Let us compare the solution so constructed with the solution of the analogous problem for flow of an ideal gas, which was studied by Guderley, Yoshihara and Barish [8 and 9], Fal'kovich and Chernov [10]. As seen from Fig.1 and Formulas (2.1), the width l, of the zone where the values of the gas parameters differ appreciably from their values in the free stream, is proportional to $r^{^2/_3}$. The difference between the local Mach number M and unity is inversely proportional to $r^{^4/_9}$, whilst the angle ϑ between the velocity vector and the axis of symmetry of the flow is inversely proportional to $r^{^5/_9}$. If, however, the coefficients of viscosity and thermal conductivity are equal to zero, then from the solution of Kármán's equations (1.11) and (1.14) it follows [8 to 10] that $l \sim r^{^4/_7}$, $M-1 \sim r^{-^6/_7}$ and $\vartheta \sim r^{-^9/_7}$. Accordingly, the influence of the dissipative factors has as a consequence the substantial blur of the region where the disturbances are concentrated and the more rapid decay of all the parameters with increasing distance from the body. The motion of a real gas is obtained without a shock wave, and it has a continuous character.

3. The problem of semi-infinite bodies. The solution constructed has an obvious interpretation, namely, it represents perturbations introduced into a stream of ideal gas which is sonic at infinity by a source placed at the origin of coordinates. As is well known, in exactly the same way one can interpret the solution describing the flow past a finite body of revolution in a stream of incompressible fluid [11]. In order to be convinced of this, let us write the expression for the flow rate Q of the gas flowing across a cylindrical control surface of radius Lr/Δ

$$Q = 2\pi\varepsilon\rho_* a_* L^2 r\int_{-\infty}^{+\infty} v_r\,dx = 2\pi\varepsilon\rho_* a_* L^2\int_{-\infty}^{+\infty} g(\xi)\,d\xi = \text{const}$$

for arbitrary values of r. From the shape of the graph of the function $g(\xi)$ it is at once obvious that $Q \neq 0$. Accordingly, the solution constructed above automatically takes account of the formation of a vortex trail behind the immersed body.

Now suppose that the flow is not past a finite body of revolution but a semi-infinite body, the profile of which is given in dimensionless variables by Equation

$$R = \varepsilon^{1/2} LA \left(\frac{x}{L}\right)^k, \qquad x \geqslant 0 \tag{3.1}$$

For the walls of this body we can take, in particular, the external surface of the boundary layer or of the vortex trail. Making use of Formulas (1.9) and omitting, as usual, primes from the dimensionless variables, let us present the boundary condition on the body under consideration in the form

$$\lim r v_r = kA^2 x^{2k-1} \quad \text{for} \quad r \to 0 \tag{3.2}$$

It is not difficult to verify that the postulated problem is self-similar and its solution is given by Formulas (2.1), but the condition (3.2) can be satisfied only if we use the integral (2.6) of Equation (2.3), according to which near the axis of symmetry

$$v_r = \frac{4a_2}{3(3n-2)} x^{-(3n-2)/2} r^{-1} + \ldots \tag{3.3}$$

Comparison of Formulas (3.2) and (3.3) gives the relation between the exponents n and k

$$n = -\,{}^4/_3 (k-1)$$

Hence it follows that when $k = 0$ the exponent of self-similarity $n = {}^4/_3$. The result obtained is natural since when $k = 0$ the problem of flow past a semi-infinite body defined by Equation (3.1) is equivalent to the problem of perturbation of a uniform stream by a source having a finite intensity.

As shown in [7], the influence of viscosity and heat conductivity can be neglected if in the equation of the profile of the semi-infinite body ${}^1/_2 < k \leqslant 1$. Then the calculation of the flow must be conducted on the basis of Kármán's system of equations. When $k = \frac{1}{2}$ the effect of the dissipative factors need not be taken into account if the numerical value of the constant A in Formula (3.1) is significantly greater than unity; conversely, when $A \ll 1$, the influence of viscosity and heat conduction becomes substantial. The solution of the complete system of equations of Liepmann-Ashkenas-Cole determines the field of the perturbations only when $k = \frac{1}{2}$ and the values of the constant A are not too different from unity. When $0 \leqslant k < {}^1/_2$ the problem of the flow past a semi-infinite body must be solved on the basis of Equations (1.11) and (1.15) derived by us, since in this case the structure of the stream is affected fundamentally by the influence of viscosity and heat conduction.

We notice that the small parameter ε in the relations (1.9) is introduced by means of Equation (3.1), stipulating the shape of the immersed semi-infinite body. When $k = 0$ and $n = {}^4/_3$, the parameter ε can be expressed in terms of the intensity Q of the source modelling the finite body of revolution together with the profile of its vortex trail. In the latter case it is appreciably more convenient to relate the quantity ε with the drag acting on the body in the sonic stream.

The drag F_x can be obtained by calculating the component of momentum along the x-axis, carried away by the perturbations in unit time across the cylindrical control surface of radius Lr/Δ. The density of flux of the x-component of momentum across the surface considered can be approximately represented in the form

$$\Pi_{xr} = \rho_* a_*^2 L^2 \varepsilon \Delta \left[v_r - \frac{1}{N_{\mathrm{Re}1}} \left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x} \right) \right] \tag{3.4}$$

As is clear from Formula (3.4), the irreversible transport of momentum from places with higher to places with lower velocities is significantly smaller in magnitude than reversible transport of momentum which is related to the mechanical motion of gas particles. Neglecting in the right-hand side of Formula (3.4) the term proportional to

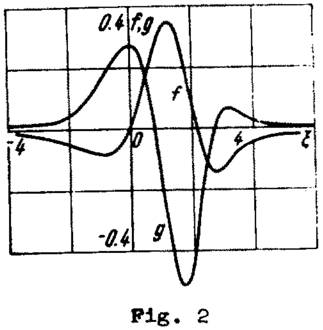

Fig. 2

$N_{\mathrm{Re}1}^{-1}$, we obtain

$$F_x = 2\pi\varepsilon\rho_* a_*^2 L^2 r \int_{-\infty}^{+\infty} v_r\, dx = a_* Q$$

The partial derivatives $\partial v_x/\partial x$ and $\partial v_r/\partial x$ satisfy the same system of equations (1.11) and (1.15) as do the particle velocity components themselves. Let us denote by $v_{x,n}(x, r)$ and $v_{r,n}(x, r)$ the general functions which have the form (2.1) with parameter n and depend on three arbitrary constants. On differentiating Expressions (2.1) with respect to x we obtain functions with the degree of similarity greater by $^2/_3$ than the original ones, i.e.

$$v_{x,n+2/3} = \frac{\partial v_{x,n}}{\partial x}, \qquad v_{r,n+2/3} = \frac{\partial v_{r,n}}{\partial x}$$

This remark can be made use of, in order to obtain the complete spectrum of eigenvalues and the corresponding eigenfunctions in the boundary value problem of finding a solution of Equation (2.3) such that when $\xi \to -\infty$ it is given by the expansion (2.5), and when $\xi \to +\infty$ it satisfies the condition (2.8).

Obviously, the boundary conditions of the problem will be satisfied if we take

$$n = \frac{2}{3}(2 + N), \quad f_n = \frac{d^N f_{4/3}^0}{d\xi^N}, \quad g_n = \frac{d^N g_{4/3}^0}{d\xi^N} \qquad (N = 0, 1, 2, \ldots) \tag{3.5}$$

where $f_{4/3}^0(\xi)$ means taking the integral (2.12) of Equation (2.3), containing one arbitrary constant, and $g_{4/3}^0(\xi)$ the corresponding function for (2.15).

As is shown by the first of Equations (3.5), the eigenvalue following after $^4/_3$ is $n = 2$. The corresponding eigenfunctions $f(\xi)$ and $g(\xi)$ are depicted in Fig.2, where it was assumed that in the expansion (2.5) the constant $a_1 = -1$. When the quantity $n = 2$ in Formulas (2.1), then the field of velocity obtained is just the same as for flow past a dipole. In fact, in thes case the moment

$$r \int_{-\infty}^{+\infty} x v_r\, dx = \int_{-\infty}^{+\infty} \xi g(\xi)\, d\xi = \mathrm{const}$$

for any value of the radius r. The last statement can be obtained also directly, by considering as an integral of the system of equations (1.11), (1.15) the linear combination of the solutions which correspond to a source and a sink, having equal intensity Q and situated on the axis of r at a distance x_0 from one another. Carrying out the limiting transition as $x_0 \to 0$, $Q \to \infty$ and $x_0 Q$ = const, we arrive at the solution (3.5) with $N = 1$.

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